Modified Gravity in Space-Time with Curvature and Torsion: $F(R,T)$ Gravity

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Abstract

In this report, we consider a theory of gravity with a metric-dependent torsion namely the $F(R,T)$ gravity, where R is the curvature scalar and T is the torsion scalar. We study the geometric root of such theory. In particular we give the derivation of the model from the geometrical point of view. Then we present the more general form of $F(R,T)$ gravity with two arbitrary functions and give some of its particular cases. In particular, the usual $F(R)$ and $F(T)$ gravity theories are particular cases of the $F(R,T)$ gravity. In the cosmological context, we find that our new gravitational theory can describe the accelerated expansion of the Universe.

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1 Introduction

In the last years the interest in modified gravity theories like $F(R)$ and $F(G)$-gravity as alternatives to the ΛCDM Model grew up. Recently, a new modified gravity theory, namely the $F(T)$-theory, has been proposed. This is a generalized version of the teleparallel gravity originally proposed by Einstein [13]-[24]. It also may describe the current cosmic acceleration without invoking dark energy. Unlike the framework of GR, where the Levi-Civita connection is used, in teleparallel gravity (TG) the used connection is the Weitzenböck'one. In principle, modification of gravity may contain a huge list of invariants and there is not any reason to restrict the gravitational

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theory to GR, TG, $F(R)$ gravity and/or $F(T)$ gravity. Indeed, several generalizations of these theories have been proposed (see e.g. the quite recent review [9]). In this paper, we study some other generalizations of $F(R)$ and $F(T)$ gravity theories. At the beginning, we briefly review the formalism of $F(R)$ gravity and $F(T)$ gravity in Friedmann-Robertson-Walker (FRW) universe. The flat FRW space-time is described by the metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{1.1}$$

where $a = a(t)$ is the scale factor. The orthonormal tetrad components $e_i(x^\mu)$ are related to the metric through

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu \,, \tag{1.2}$$

where the Latin indices i, j run over 0...3 for the tangent space of the manifold, while the Greek letters μ, ν are the coordinate indices on the manifold, also running over 0...3.

$F(R)$ and $F(T)$ modified theories of gravity have been extensively explored and the possibility to construct viable models in their frameworks has been carefully analyzed in several papers (see [9] for a recent review). For such theories, the physical motivations are principally related to the possibility to reach a more realistic representation of the gravitational fields near curvature singularities and to create some first order approximation for the quantum theory of gravitational fields. Recently, it has been registred a renaissance of $F(R)$ and $F(T)$ gravity theories in the attempt to explain the late-time accelerated expansion of the Universe [9].

PROBLEM:

Construct such $F(R,T)$ gravity which contents $F(R)$ gravity and $F(T)$ gravity as particular cases.

In the modern cosmology, in order to construct (generalized) gravity theories, three quantities – the curvature scalar, the Gauss –Bonnet scalar and the torsion scalar – are usually used (about our notations see below):

$$\begin{aligned} R_s &= g^{\mu\nu} R_{\mu\nu}, & (1.3)\\ G_s &= R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, & (1.4)\\ T_s &= S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}. & (1.5) \end{aligned}$$

In this paper, our aim is to replace these quantities with the other three variables in the form

$$\begin{aligned} R &= u + g^{\mu\nu} R_{\mu\nu}, & (1.6)\\ G &= w + R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, & (1.7)\\ T &= v + S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, & (1.8) \end{aligned}$$

where $u = u(x_i; g_{ij}, \dot{g_{ij}}, \ddot{g_{ij}}, ...; f_j)$, $v = v(x_i; g_{ij}, \dot{g_{ij}}, \ddot{g_{ij}}, ...; g_j)$ and $w = w(x_i; g_{ij}, \dot{g_{ij}}, \ddot{g_{ij}}, ...; h_j)$ are some functions to be defined. As a result, we obtain some generalizations of the known modified gravity theories. With the FRW metric ansatz the three variables (1.3)-(1.5) become

$$\begin{aligned} R_s &= 6(\dot{H} + 2H^2), & (1.9)\\ G_s &= 24H^2(\dot{H} + H^2), & (1.10)\\ T_s &= -6H^2, & (1.11) \end{aligned}$$

where $H = (\ln a)_t$. In the contrast, in this paper we will use the following three variables

$$\begin{aligned} R &= u + 6(\dot{H} + 2H^2), & (1.12)\\ G &= w + 24H^2(\dot{H} + H^2), & (1.13)\\ T &= v - 6H^2. & (1.14) \end{aligned}$$

Finally we would like to note that we expect $w = w(u, v)$. This paper is organized as follows. In Sec. 2, we briefly review the formalism of $F(R)$ and $F(T)$-gravity for FRW metric. In particular, the corresponding Lagrangians are explicitly presented. In Sec. 3, we consider $F(R,T)$ theory, where R and T will be generalized with respect to the usual notions of curvature scalar and torsion scalar. Some reductions of $F(R,T)$ gravity are presented in Sec. 4. In Sec. 5, the specific model $F(R,T) = \mu R + \nu T$ is analized and in Sec. 6 the exact power-law solution is found; some cosmological implications of the model will be here discussed. The Bianchi type I version of $F(R,T)$ gravity is considered in Sec. 7. Sec. 8 is devoted to some generalizations of some modified gravity theories. Final conclusions and remarks are provided in Sec. 9.

2 Preliminaries of $F(R)$, $F(G)$ and $F(T)$ gravities

At the beginning, we present the basic equations of $F(R)$, $F(T)$ and $F(G)$ modified gravity theories. For simplicity we mainly work in the FRW spacetime.

2.1 $F(R)$ gravity

The action of $F(R)$ theory is given by

$$\mathcal{S}_R = \int d^4xe[F(R) + L_m], \tag{2.1}$$

where R is the curvature scalar. We work with the FRW metric (1.1). In this case R assumes the form

$$R = R_s = 6(\dot{H} + 2H^2). \tag{2.2}$$

The action we rewrite as

$$\mathcal{S}_R = \int dtL_R, \tag{2.3}$$

where the Lagrangian is given by

$$L_R = a^3(F - RF_R) - 6F_R a\dot{a}^2 - 6F_{RR}\dot{R}a^2\dot{a} - a^3L_m. \tag{2.4}$$

The corresponding field equations of $F(R)$ gravity read

$$\begin{aligned} 6\dot{R}HF_{RR} - (R - 6H^2)F_R + F &= \rho, & (2.5)\\ -2\dot{R}^2F_{RRR} + [-4\dot{R}H - 2\ddot{R}]F_{RR} + [-2H^2 - 4a^{-1}\ddot{a} + R]F_R - F &= p, & (2.6)\\ \dot{\rho} + 3H(\rho + p) &= 0. & (2.7)\end{aligned}$$

2.2 $F(T)$ gravity

In the modified teleparallel gravity, the gravitational action is

$$\mathcal{S}_T = \int d^4xe[F(T) + L_m], \tag{2.8}$$

where $e = \det(e^i_\mu) = \sqrt{-g}$, and for convenience we use the units $16\pi G = \hbar = c = 1$ throughout. The torsion scalar T is defined as

$$T \equiv S_\rho{}^{\mu\nu}\, T^\rho{}_{\mu\nu}\,, \tag{2.9}$$

where

$$\begin{aligned} T^\rho{}_{\mu\nu} &\equiv -e^\rho_i\left(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu\right), & (2.10)\\ K^{\mu\nu}{}_\rho &\equiv -\frac{1}{2}\left(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}\right), & (2.11)\\ S_\rho{}^{\mu\nu} &\equiv \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta\right). & (2.12)\end{aligned}$$

For a spatially flat FRW metric (1.1), as a consequence of equations (3.9) and (1.1), we have that the torsion scalar assumes the form

$$T = T_s = -6H^2. \tag{2.13}$$

The action (3.8) can be written as

$$\mathcal{S}_T = \int dt L_T, \tag{2.14}$$

where the point-like Lagrangian reads

$$L_T = a^3 (F - F_T T) - 6F_T a\dot{a}^2 - a^3 L_m. \tag{2.15}$$

The equations of F(T) gravity look like

$$\begin{aligned} 12H^2 F_T + F &= \rho, && (2.16) \\ 48H^2 F_{TT}\dot{H} - F_T \left(12H^2 + 4\dot{H}\right) - F &= p, && (2.17) \\ \dot{\rho} + 3H(\rho + p) &= 0. && (2.18) \end{aligned}$$

2.3 $F(G)$ gravity

The action of $F(G)$ theory is given by

$$\mathcal{S}_G = \int d^4 x e[F(G) + L_m], \tag{2.19}$$

where the Gauss – Bonnet scalar G for the FRW metric is

$$G = G_s = 24H^2(\dot{H} + H^2). \tag{2.20}$$

3 Geometrical roots of $F(R,T)$ gravity

We start from the M_{43} - model (about our notations, see e.g. Refs. [10]-[11]). This model is one of the representatives of $F(R,T)$ gravity. The action of the M_{43} - model reads as

$$\begin{aligned} S_{43} &= \int d^4 x \sqrt{-g}[F(R,T) + L_m], \\ R &= R_s = \epsilon_1 g^{\mu\nu} R_{\mu\nu}, \\ T &= T_s = \epsilon_2 S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \end{aligned} \tag{3.1}$$

where L_m is the matter Lagrangian, $\epsilon_i = \pm 1$ (signature), R is the curvature scalar, T is the torsion scalar (about our notation see below). In this section we try to give one of the possible geometric formulations of M_{43} - model. Note that we have different cases related with the signature: (1) $\epsilon_1 = 1, \epsilon_2 = 1$; (2) $\epsilon_1 = 1, \epsilon_2 = -1$; (3) $\epsilon_1 = -1, \epsilon_2 = 1$; (4) $\epsilon_1 = -1, \epsilon_2 = -1$. Also note that M_{43} - model is a particular case of M_{37} - model having the form

$$\begin{aligned} S_{37} &= \int d^4 x \sqrt{-g}[F(R,T) + L_m], \\ R &= u + R_s = u + \epsilon_1 g^{\mu\nu} R_{\mu\nu}, \\ T &= v + T_s = v + \epsilon_2 S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \end{aligned} \tag{3.2}$$

where

$$R_s = \epsilon_1 g^{\mu\nu} R_{\mu\nu}, \quad T_s = \epsilon_2 S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} \tag{3.3}$$

are the standard forms of the curvature and torsion scalars.

3.1 General case

To understand the geometry of the M_{43} - model, we consider some spacetime with the curvature and torsion so that its connection $G^\lambda{}_{\mu\nu}$ is a sum of the curvature and torsion parts. In this paper, the Greek alphabets ($\mu,\ \nu,\ \rho,\ ... = 0,1,2,3$) are related to spacetime, and the Latin alphabets

$(i, j, k, ... = 0, 1, 2, 3)$ denote indices, which are raised and lowered with the Minkowski metric η_{ij} = diag $(-1, +1, +1, +1)$. For our spacetime the connection $G^{\lambda}{}_{\mu\nu}$ has the form

$$G^{\lambda}{}_{\mu\nu} = e_i{}^{\lambda}\partial_{\mu}e^i{}_{\nu} + e_j{}^{\lambda}e^i{}_{\nu}\omega^j{}_{i\mu} = \Gamma^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}. \tag{3.4}$$

Here $\Gamma^{j}_{i\mu}$ is the Levi-Civita connection and $K^{j}_{i\mu}$ is the contorsion. Let the metric has the form

$$ds^2 = g_{ij}dx^i dx^j. \tag{3.5}$$

Then the orthonormal tetrad components $e_i(x^{\mu})$ are related to the metric through

$$g_{\mu\nu} = \eta_{ij}e^i_{\mu}e^j_{\nu}, \tag{3.6}$$

so that the orthonormal condition reads as

$$\eta_{ij} = g_{\mu\nu}e^{\mu}_i e^{\nu}_j. \tag{3.7}$$

Here $\eta_{ij} = diag(-1, 1, 1, 1)$, where we used the relation

$$e^i_{\mu}e^{\mu}_j = \delta^i_j. \tag{3.8}$$

The quantities $\Gamma^{j}_{i\mu}$ and $K^{j}_{i\mu}$ are defined as

$$\Gamma^{l}{}_{jk} = \frac{1}{2}g^{lr}\{\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}\} \tag{3.9}$$

and

$$K^{\lambda}_{\mu\nu} = -\frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} + T_{\mu\nu}{}^{\lambda} + T_{\nu\mu}{}^{\lambda}\right), \tag{3.10}$$

respectively. Here the components of the torsion tensor are given by

$$\begin{aligned} T^{\lambda}{}_{\mu\nu} &= e_i{}^{\lambda}T^i{}_{\mu\nu} = G^{\lambda}{}_{\mu\nu} - G^{\lambda}{}_{\nu\mu}, & (3.11)\\ T^i{}_{\mu\nu} &= \partial_{\mu}e^i{}_{\nu} - \partial_{\nu}e^i{}_{\mu} + G^i{}_{j\mu}e^j{}_{\nu} - G^i{}_{j\nu}e^j{}_{\mu}. & (3.12)\end{aligned}$$

The curvature $R^{\rho}{}_{\sigma\mu\nu}$ is defined as

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= e_i{}^{\rho}e^j{}_{\sigma}R^i{}_{j\mu\nu} = \partial_{\mu}G^{\rho}{}_{\sigma\nu} - \partial_{\nu}G^{\rho}{}_{\sigma\mu} + G^{\rho}{}_{\lambda\mu}G^{\lambda}{}_{\sigma\nu} - G^{\rho}{}_{\lambda\nu}G^{\lambda}{}_{\sigma\mu} \\ &= \bar{R}^{\rho}{}_{\sigma\mu\nu} + \partial_{\mu}K^{\rho}{}_{\sigma\nu} - \partial_{\nu}K^{\rho}{}_{\sigma\mu} + K^{\rho}{}_{\lambda\mu}K^{\lambda}{}_{\sigma\nu} - K^{\rho}{}_{\lambda\nu}K^{\lambda}{}_{\sigma\mu} \\ &\quad +\Gamma^{\rho}_{\lambda\mu}K^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}_{\lambda\nu}K^{\lambda}{}_{\sigma\mu} + \Gamma^{\lambda}_{\sigma\nu}K^{\rho}{}_{\lambda\mu} - \Gamma^{\lambda}_{\sigma\mu}K^{\rho}{}_{\lambda\nu}, \end{aligned} \tag{3.13}$$

where the Riemann curvature of the Levi-Civita connection is defined in the standard way

$$\bar{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\sigma\mu}. \tag{3.14}$$

Now we introduce two important quantities namely the curvature (R) and torsion (T) scalars as

$$\begin{aligned} R &= g^{ij}R_{ij}, & (3.15)\\ T &= S^{\mu\nu}_{\rho}T^{\rho}_{\mu\nu}, & (3.16)\end{aligned}$$

where

$$S_{\rho}{}^{\mu\nu} = \frac{1}{2}\left(K_{\rho}{}^{\mu\nu} + \delta^{\mu}_{\rho}T_{\theta}{}^{\theta\nu} - \delta^{\nu}_{\rho}T_{\theta}{}^{\theta\mu}\right). \tag{3.17}$$

Then the M_{43} - model we write in the form (3.1). To conclude this subsection, we note that in GR, it is postulated that $T^{\lambda}{}_{\mu\nu} = 0$ and such 4-dimensional spacetime manifolds with metric and without torsion are labelled as V_4. At the same time, it is a general convention to call U_4, the manifolds endowed with metric and torsion.

3.2 FRW case

From here we work with the spatially flat FRW metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{3.18}$$

where $a(t)$ is the scale factor. In this case, the non-vanishing components of the Levi-Civita connection are

$$\begin{aligned} \Gamma^0_{00} &= \Gamma^0_{0i} = \Gamma^0_{i0} = \Gamma^i_{00} = \Gamma^i_{jk} = 0, \\ \Gamma^0_{ij} &= a^2 H \delta_{ij}, \\ \Gamma^i_{jo} &= \Gamma^i_{0j} = H\delta^i_j, \end{aligned} \tag{3.19}$$

where $H = (\ln a)_t$ and $i, j, k, ... = 1, 2, 3$. Now let us calculate the components of torsion tensor. Its non-vanishing components are given by:

$$\begin{aligned} T_{110} &= T_{220} = T_{330} = a^2 h, \\ T_{123} &= T_{231} = T_{312} = 2a^3 f, \end{aligned} \tag{3.20}$$

where h and f are some real functions (see e.g. Refs. [12]). Note that the indices of the torsion tensor are raised and lowered with respect to the metric, that is

$$T_{ijk} = g_{kl} T_{ij}{}^l. \tag{3.21}$$

Now we can find the contortion components. We get

$$\begin{aligned} K^1{}_{10} &= K^2{}_{20} = K^3{}_{30} = 0, \\ K^1{}_{01} &= K^2{}_{02} = K^3{}_{03} = h, \\ K^0{}_{11} &= K^0{}_{22} = K^0{}_{22} = a^2 h, \\ K^1{}_{23} &= K^2{}_{31} = K^3{}_{12} = -af, \\ K^1{}_{32} &= K^2{}_{13} = K^3{}_{21} = af. \end{aligned} \tag{3.22}$$

The non-vanishing components of the curvature $R^\rho{}_{\sigma\mu\nu}$ are given by

$$\begin{aligned} R^0{}_{101} &= R^0{}_{202} = R^0{}_{303} = a^2(\dot{H} + H^2 + Hh + \dot{h}), \\ R^0{}_{123} &= -R^0{}_{213} = R^0{}_{312} = 2a^3 f(H + h), \\ R^1{}_{203} &= -R^1{}_{302} = R^2{}_{301} = -a(Hf + \dot{f}), \\ R^1{}_{212} &= R^1{}_{313} = R^2{}_{323} = a^2[(H + h)^2 - f^2]. \end{aligned} \tag{3.23}$$

Similarly, we write the non-vanishing components of the Ricci curvature tensor $R_{\mu\nu}$ as

$$\begin{aligned} R_{00} &= -3\dot{H} - 3\dot{h} - 3H^2 - 3Hh, \\ R_{11} &= R_{22} = R_{33} = a^2(\dot{H} + \dot{h} + 3H^2 + 5Hh + 2h^2 - f^2). \end{aligned} \tag{3.24}$$

At the same time, the non-vanishing components of the tensor $S_\rho^{\mu\nu}$ are given by

$$S_1^{10} = \frac{1}{2}\left(K_1^{10} + \delta_1^1 T_\theta^{\theta 0} - \delta_1^0 T_\theta^{\theta\nu}\right) = \frac{1}{2}(h + 2h) = h, \tag{3.25}$$

$$S_1^{10} = S_2^{20} = S_3^{30} = 2h, \tag{3.26}$$

$$S_1^{23} = \frac{1}{2}\left(K_1^{23} + \delta_1^2 + \delta_1^3\right) = -\frac{f}{2a}, \tag{3.27}$$

$$S_1^{23} = S_2^{31} = S_3^{21} = -\frac{f}{2a} \tag{3.28}$$

and

$$T = T^1_{10} S_1^{10} + T^2_{20} S_2^{20} + T^3_{30} S_3^{30} + T_1^{23} S^1_{23} + T^2_{31} S_2^{31} + T^3_{12} S_3^{12}. \tag{3.29}$$

Now we are ready to write the explicit forms of the curvature and torsion scalars. We have

$$
\begin{aligned}
R &= 6(\dot{H}+2H^2)+6\dot{h}+18Hh+6h^2-3f^2 && (3.30)\\
T &= 6(h^2-f^2). && (3.31)
\end{aligned}
$$

So finally for the FRW metric, the M_{43} - model takes the form

$$
\begin{aligned}
S_{43} &= \int d^4x\sqrt{-g}[F(R,T)+L_m],\\
R &= 6(\dot{H}+2H^2)+6\dot{h}+18Hh+6h^2-3f^2,\\
T &= 6(h^2-f^2).
\end{aligned}
\tag{3.32}
$$

It (that is the M_{43} - model) is one of geometrical realizations of $F(R,T)$ gravity in the sense that it was derived from the purely geometrical point of view.

4 Lagrangian formulation of $F(R,T)$ gravity

Of course, we can work with the form (3.32) of $F(R,T)$ gravity. But a more interesting and general form of $F(R,T)$ gravity is the so-called M_{37} - model. The action of the M_{37} - gravity reads as [10]

$$
\begin{aligned}
S_{37} &= \int d^4x\sqrt{-g}[F(R,T)+L_m],\\
R &= u+R_s = u+6\epsilon_1(\dot{H}+2H^2),\\
T &= v+T_s = v+6\epsilon_2 H^2,
\end{aligned}
\tag{4.1}
$$

where

$$
R_s = 6\epsilon_1(\dot{H}+2H^2), \quad T_s = 6\epsilon_2 H^2. \tag{4.2}
$$

So we can see that here instead of two functions h and f in (3.32), we introduced two new functions u and v. For example, for (3.32) these functions have the form

$$
\begin{aligned}
u &= 6(1-\epsilon_1)(\dot{H}+2H^2)+6\dot{h}+18Hh+6h^2-3f^2, && (4.3)\\
v &= 6(h^2-f^2-\epsilon_2 H^2) && (4.4)
\end{aligned}
$$

that again tells us that the M_{43} - model is a particular case of M_{37} - model [Note that if $\epsilon_1 = 1 = \epsilon_2$ we have $u = 6\dot{h}+18Hh+6h^2-3f^2, \quad v = 6(h^2-f^2-H^2)$]. But in general we think (or assume) that $u = u(t,a,\dot{a},\ddot{a},\dddot{a},...;f_i)$ and $v = v(t,a,\dot{a},\ddot{a},\dddot{a},...;g_i)$, while f_i and g_i are some unknown functions related with the geometry of the spacetime. So below we will work with a more general form of $F(R,T)$ gravity namely the M_{37} - gravity (4.1). Introducing the Lagrangian multipliers we now can rewrite the action (4.1) as

$$
S_{37} = \int dt\, a^3\left\{F(R,T)-\lambda\left[T-v-6\epsilon_2\frac{\dot{a}^2}{a^2}\right]-\gamma\left[R-u-6\epsilon_1\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right]+L_m\right\}, \tag{4.5}
$$

where λ and γ are Lagrange multipliers. If we take the variations with respect to T and R of this action, we get

$$
\lambda = F_T, \quad \gamma = F_R. \tag{4.6}
$$

Therefore, the action (4.5) can be rewritten as

$$
S_{37} = \int dt\, a^3\left\{F(R,T)-F_T\left[T-v-6\epsilon_2\frac{\dot{a}^2}{a^2}\right]-F_R\left[R-u-6\epsilon_1\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right]+L_m\right\}. \tag{4.7}
$$

Then the corresponding point-like Lagrangian reads

$$
L_{37} = a^3[F-(T-v)F_T-(R-u)F_R+L_m]-6(\epsilon_1F_R-\epsilon_2F_T)a\dot{a}^2-6\epsilon_1(F_{RR}\dot{R}+F_{RT}\dot{T})a^2\dot{a}. \tag{4.8}
$$

As is well known, for our dynamical system, the Euler-Lagrange equations read as

$$\frac{d}{dt}\left(\frac{\partial L_{37}}{\partial \dot{R}}\right)-\frac{\partial L_{37}}{\partial R} = 0, \tag{4.9}$$

$$\frac{d}{dt}\left(\frac{\partial L_{37}}{\partial \dot{T}}\right)-\frac{\partial L_{37}}{\partial T} = 0, \tag{4.10}$$

$$\frac{d}{dt}\left(\frac{\partial L_{37}}{\partial \dot{a}}\right)-\frac{\partial L_{37}}{\partial a} = 0. \tag{4.11}$$

Hence, using the expressions

$$\frac{\partial L_{37}}{\partial \dot{R}} = -6\epsilon_1 F_{RR}a^2\dot{a}, \tag{4.12}$$

$$\frac{\partial L_{37}}{\partial \dot{T}} = -6\epsilon_1 F_{RT}a^2\dot{a}, \tag{4.13}$$

$$\frac{\partial L_{37}}{\partial \dot{a}} = -12(\epsilon_1 F_R-\epsilon_2 F_T)a\dot{a}-6\epsilon_1(F_{RR}\dot{R}+F_{RT}\dot{T}+F_{R\psi}\dot{\psi})a^2+a^3F_Tv_{\dot{a}}+a^3F_Ru_{\dot{a}}, \tag{4.14}$$

we get

$$a^3F_{TT}\left(T-v-6\epsilon_2\frac{\dot{a}^2}{a^2}\right) = 0, \tag{4.15}$$

$$a^3F_{RR}\left(R-u-6\epsilon_1(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2})\right) = 0, \tag{4.16}$$

$$U+B_2F_{TT}+B_1F_T+C_2F_{RRT}+C_1F_{RTT}+C_0F_{RT}+MF+6\epsilon_2a^2p = 0, \tag{4.17}$$

respectively. Here

$$U = A_3F_{RRR}+A_2F_{RR}+A_1F_R, \tag{4.18}$$

$$A_3 = -6\epsilon_1\dot{R}^2a^2, \tag{4.19}$$

$$A_2 = -12\epsilon_1\dot{R}a\dot{a}-6\epsilon_1\ddot{R}a^2+a^3\dot{R}u_{\dot{a}}, \tag{4.20}$$

$$A_1 = 12\epsilon_1\dot{a}^2+6\epsilon_1a\ddot{a}+3a^2\dot{a}u_{\dot{a}}+a^3\dot{u}_{\dot{a}}-a^3u_a, \tag{4.21}$$

$$B_2 = 12\epsilon_2\dot{T}a\dot{a}+a^3\dot{T}v_{\dot{a}}, \tag{4.22}$$

$$B_1 = 24\epsilon_2\dot{a}^2+12\epsilon_2a\ddot{a}+3a^2\dot{a}v_{\dot{a}}+a^3\dot{v}_{\dot{a}}-a^3v_a, \tag{4.23}$$

$$C_2 = -12\epsilon_1a^2\dot{R}\dot{T}, \tag{4.24}$$

$$C_1 = -6\epsilon_1a^2\dot{T}^2, \tag{4.25}$$

$$C_0 = -12\epsilon_1\dot{T}a\dot{a}+12\epsilon_2\dot{R}a\dot{a}-6\epsilon_1a^2\ddot{T}+a^3\dot{R}v_{\dot{a}}+a^3\dot{T}u_{\dot{a}}, \tag{4.26}$$

$$M = -3a^2. \tag{4.27}$$

If $F_{RR}\neq 0, \quad F_{TT}\neq 0$, from Eqs. (4.17) and (4.17), it is easy to find that

$$R=u+6\epsilon_1(\dot{H}+2H^2), \quad T=v+6\epsilon_2H^2, \tag{4.28}$$

so that the relations (4.1) are recovered. Generally, these equations are the Euler constraints of the dynamics. Here a, R, T are the generalized coordinates of the configuration space. On the other hand, it is also well known that the total energy (Hamiltonian) corresponding to Lagrangian L_{37} is given by

$$H_{37}=\frac{\partial L_{37}}{\partial \dot{a}}\dot{a}+\frac{\partial L_{37}}{\partial \dot{R}}\dot{R}+\frac{\partial L_{37}}{\partial \dot{T}}\dot{T}-L_{37}. \tag{4.29}$$

Hence using (4.12)-(4.14) we obtain

$$\begin{aligned} H_{37}=[&-12(\epsilon_1F_R-\epsilon_2F_T)a\dot{a}-6\epsilon_1(F_{RR}\dot{R}+F_{RT}\dot{T}+F_{R\psi}\dot{\psi})a^2+a^3F_Tv_{\dot{a}}+a^3F_Ru_{\dot{a}}]\dot{a}\\ &-6\epsilon_1F_{RR}a^2\dot{a}\dot{R}-6\epsilon_1F_{RT}a^2\dot{a}\dot{T}-[a^3(F-TF_T-RF_R+vF_T+uF_R+L_m)-\\ &6(\epsilon_1F_R-\epsilon_2F_T)a\dot{a}^2-6\epsilon_1(F_{RR}\dot{R}+F_{RT}\dot{T}+F_{R\psi}\dot{\psi})a^2\dot{a}]. \end{aligned} \tag{4.30}$$

Let us rewrite this formula as

$$H_{37} = D_2F_{RR} + D_1F_R + JF_{RT} + E_1F_T + KF + 2a^3\rho, \tag{4.31}$$

where

$$\begin{aligned} D_2 &= -6\epsilon_1\dot{R}a^2\dot{a}, && (4.32)\\ D_1 &= 6\epsilon_1 a\ddot{a} + a^3u_{\dot{a}}\dot{a}, && (4.33)\\ J &= -6\epsilon_1 a^2\dot{a}\dot{T}, && (4.34)\\ E_1 &= 12\epsilon_2 a\dot{a}^2 + a^3v_{\dot{a}}\dot{a}, && (4.35)\\ K &= -a^3. && (4.36) \end{aligned}$$

As usual we assume that the total energy $H_{37} = 0$ (Hamiltonian constraint). So finally we have the following equations of the M_{37} - model [10]-[11]:

$$\begin{aligned} D_2F_{RR} + D_1F_R + JF_{RT} + E_1F_T + KF &= -2a^3\rho,\\ U + B_2F_{TT} + B_1F_T + C_2F_{RRT} + C_1F_{RTT} + C_0F_{RT} + MF &= 6a^2p,\\ \dot{\rho} + 3H(\rho+p) &= 0. \end{aligned} \tag{4.37}$$

It deserves to note that the M_{37} - model (4.1) admits some interesting particular and physically important cases. Some particular cases are now presented.

i) The M_{44} - model. Let the function $F(R,T)$ be independent from the torsion scalar T: $F = F(R,T) = F(R)$. Then the action (4.1) acquires the form

$$S_{44} = \int d^4xe[F(R) + L_m], \tag{4.38}$$

where

$$R = u + R_s = u + \epsilon_1 g^{\mu\nu}R_{\mu\nu}, \tag{4.39}$$

is the curvature scalar. It is the M_{44} - model. We work with the FRW metric. In this case R takes the form

$$R = u + 6\epsilon_1(\dot{H} + 2H^2). \tag{4.40}$$

The action can be rewritten as

$$S_{44} = \int dtL_{44}, \tag{4.41}$$

where the Lagrangian is given by

$$L_{44} = a^3[F - (R-u)F_R + L_m] - 6\epsilon_1F_Ra\dot{a}^2 - 6\epsilon_1F_{RR}\dot{R}a^2\dot{a}. \tag{4.42}$$

The corresponding field equations of the M_{44} - model read as

$$\begin{aligned} D_2F_{RR} + D_1F_R + KF &= -2a^3\rho,\\ A_3F_{RRR} + A_2F_{RR} + A_1F_R + MF &= 6a^2p,\\ \dot{\rho} + 3H(\rho+p) &= 0. \end{aligned} \tag{4.43}$$

Here

$$\begin{aligned} D_2 &= -6\epsilon_1\dot{R}a^2\dot{a}, && (4.44)\\ D_1 &= 6\epsilon_1 a^2\ddot{a} + a^3u_{\dot{a}}\dot{a}, && (4.45)\\ K &= -a^3 && (4.46) \end{aligned}$$

and

$$\begin{aligned} A_3 &= -6\epsilon_1\dot{R}^2a^2, && (4.47)\\ A_2 &= -12\epsilon_1\dot{R}a\dot{a} - 6\epsilon_1\ddot{R}a^2 + a^3\dot{R}u_{\dot{a}}, && (4.48)\\ A_1 &= 12\epsilon_1\dot{a}^2 + 6\epsilon_1 a\ddot{a} + 3a^2\dot{a}u_{\dot{a}} + a^3\dot{u}_{\dot{a}} - a^3u_a, && (4.49)\\ M &= -3a^2. && (4.50) \end{aligned}$$

If $u=0$ then we get the following equations of the standard $F(R_s)$ gravity (after $R=R_s$):

$$\begin{aligned} 6\dot{R}HF_{RR}-(R-6H^2)F_R+F &= \rho, && (4.51)\\ -2\dot{R}^2F_{RRR}+[-4\dot{R}H-2\ddot{R}]F_{RR}+[-2H^2-4a^{-1}\ddot{a}+R]F_R-F &= p, && (4.52)\\ \dot{\rho}+3H(\rho+p) &= 0. && (4.53)\end{aligned}$$

ii) The M_{45} - model. The action of the M_{45} - model looks like

$$S_{45}=\int d^4xe[F(T)+L_m], \tag{4.54}$$

where $e=\det\left(e^i_\mu\right)=\sqrt{-g}$ and the torsion scalar T is defined as

$$T=v+T_s=v+\epsilon_2 S_\rho{}^{\mu\nu}T^\rho{}_{\mu\nu}. \tag{4.55}$$

Here

$$\begin{aligned} T^\rho{}_{\mu\nu} &\equiv -e^\rho_i\left(\partial_\mu e^i_\nu-\partial_\nu e^i_\mu\right), && (4.56)\\ K^{\mu\nu}{}_\rho &\equiv -\frac{1}{2}\left(T^{\mu\nu}{}_\rho-T^{\nu\mu}{}_\rho-T_\rho{}^{\mu\nu}\right), && (4.57)\\ S_\rho{}^{\mu\nu} &\equiv \frac{1}{2}\left(K^{\mu\nu}{}_\rho+\delta^\mu_\rho T^{\theta\nu}{}_\theta-\delta^\nu_\rho T^{\theta\mu}{}_\theta\right). && (4.58)\end{aligned}$$

For a spatially flat FRW metric (3.18), we have the torsion scalar in the form

$$T=v+T_s=v+6\epsilon_2 H^2. \tag{4.59}$$

The action (4.54) can be written as

$$S_{45}=\int dtL_{45}, \tag{4.60}$$

where the point-like Lagrangian reads

$$L_{45}=a^3[F-(T-v)F_T+L_m]+6\epsilon_2F_Ta\dot{a}^2. \tag{4.61}$$

So finally we get the following equations of the M_{45} - model:

$$\begin{aligned} E_1F_T+KF &= -2a^3\rho,\\ B_2F_{TT}+B_1F_T+MF &= 6a^2p, && (4.62)\\ \dot{\rho}+3H(\rho+p) &= 0.\end{aligned}$$

Here

$$\begin{aligned} E_1 &= 12\epsilon_2a\dot{a}^2+a^3v_{\dot{a}}\dot{a}, && (4.63)\\ K &= -a^3 && (4.64)\end{aligned}$$

and

$$\begin{aligned} B_2 &= 12\epsilon_2\dot{T}a\dot{a}+a^3\dot{T}v_{\dot{a}}, && (4.65)\\ B_1 &= 24\epsilon_2\dot{a}^2+12\epsilon_2a\ddot{a}+3a^2\dot{a}v_{\dot{a}}+a^3\dot{v}_{\dot{a}}-a^3v_a, && (4.66)\\ M &= -3a^2. && (4.67)\end{aligned}$$

If we put $v=0$ then the M_{45} - model reduces to the usual $F(T_s)$ gravity, where $T_s=6\epsilon_2H^2$. As is well-known the equations of $F(T_s)$ gravity are given by

$$\begin{aligned} 12H^2F_T+F &= \rho, && (4.68)\\ 48H^2F_{TT}\dot{H}-F_T\left(12H^2+4\dot{H}\right)-F &= p, && (4.69)\\ \dot{\rho}+3H(\rho+p) &= 0, && (4.70)\end{aligned}$$

where we must put $T=T_s$. Finally we note that it is well-known that the standard $F(T_s)$ gravity is not local Lorentz invariant [33]. In this context, we have a very meager hope that the M_{45} - model (4.54) is free from such problems.

5 Cosmological solutions

In this section we investigate cosmological consequences of the $F(R,T)$ gravity. As example, we want to find some exact cosmological solutions of the M_{37} - gravity model. Since its equations are very complicated we here consider the simplest case when

$$F(R,T) = \mu R + \nu T, \tag{5.1}$$

where μ and ν are some constants. Then equations (4.37) take the form

$$\begin{aligned} \mu D_1 + \nu E_1 + K(\nu T + \mu R) &= -2a^3\rho, \\ \mu A_1 + \nu B_1 + M(\nu T + \mu R) &= 6a^2 p, \\ \dot{\rho} + 3H(\rho + p) &= 0, \end{aligned} \tag{5.2}$$

where

$$D_1 = 6\epsilon_1 a^2\ddot{a} + a^3 u_{\dot{a}}\dot{a}, \tag{5.3}$$
$$E_1 = 12\epsilon_2 a\dot{a}^2 + a^3 v_{\dot{a}}\dot{a}, \tag{5.4}$$
$$K = -a^3, \tag{5.5}$$
$$A_1 = 12\epsilon_1\dot{a}^2 + 6\epsilon_1 a\ddot{a} + 3a^2\dot{a}u_{\dot{a}} + a^3\dot{u}_{\dot{a}} - a^3 u_a, \tag{5.6}$$
$$B_1 = 24\epsilon_2\dot{a}^2 + 12\epsilon_2 a\ddot{a} + 3a^2\dot{a}v_{\dot{a}} + a^3\dot{v}_{\dot{a}} - a^3 v_a, \tag{5.7}$$
$$M = -3a^2. \tag{5.8}$$

We can rewrite this system as

$$\begin{aligned} 3\sigma H^2 - 0.5(\dot{a}z_{\dot{a}} - z) &= \rho, \\ -\sigma(2\dot{H} + 3H^2) + 0.5(\dot{a}z_{\dot{a}} - z) + \frac{1}{6}a(\dot{z}_{\dot{a}} - z_a) &= p, \\ \dot{\rho} + 3H(\rho + p) &= 0, \end{aligned} \tag{5.9}$$

where $z = \mu u + \nu v, \quad \sigma = \mu\epsilon_1 - \nu\epsilon_2$. This system contents two independent equations for five unknown functions (a, ρ, p, u, v). But in fact it contain 4 unknown functions (H, ρ, p, z). The corresponding EoS parameter reads as

$$\omega = \frac{p}{\rho} = -1 + \frac{-2\sigma\dot{H} + \frac{1}{6}a(\dot{z}_{\dot{a}} - z_a)}{3\sigma H^2 - 0.5(\dot{a}z_{\dot{a}} - z)}. \tag{5.10}$$

Let us find some simplest cosmological solutions of the system (5.9).

5.1 Example 1

We start from the case $\sigma = 0$. In this case the system (5.9) takes the form

$$\begin{aligned} -0.5(\dot{a}z_{\dot{a}} - z) &= \rho, \\ 0.5(\dot{a}z_{\dot{a}} - z) + \frac{1}{6}a(\dot{z}_{\dot{a}} - z_a) &= p, \\ \dot{\rho} + 3H(\rho + p) &= 0. \end{aligned} \tag{5.11}$$

At the same time the EoS parameter becomes

$$\omega = \frac{p}{\rho} = -1 - \frac{a(\dot{z}_{\dot{a}} - z_a)}{3(\dot{a}z_{\dot{a}} - z)}. \tag{5.12}$$

Now we assume that

$$z = \kappa a^l, \tag{5.13}$$

where κ and l are some real constants. Then

$$\omega = -1 - \frac{l}{3}. \tag{5.14}$$

This result tells us that in this case our model can describes the accelerated expansion of the Universe for some values of l.

5.2 Example 2

Now consider the de Sitter case that is $H = H_0 = const$ so that $a = a_0 e^{H_0 t}$. Then the system (5.9) reads as

$$
\begin{aligned}
3\sigma H_0^2 - 0.5(\dot{a}z_{\dot{a}} - z) &= \rho, \\
-3\sigma H_0^2 + 0.5(\dot{a}z_{\dot{a}} - z) + \frac{1}{6}a(\dot{z}_{\dot{a}} - z_a) &= p, \\
\dot{\rho} + 3H(\rho + p) &= 0.
\end{aligned} \tag{5.15}
$$

The EoS parameter takes the form

$$\omega = \frac{p}{\rho} = -1 + \frac{a(\dot{z}_{\dot{a}} - z_a)}{18\sigma H_0^2 - 3(\dot{a}z_{\dot{a}} - z)}. \tag{5.16}$$

If z has the form (5.13) that is $z = \kappa e^{H_0 l t}$ then we have

$$\omega = \frac{p}{\rho} = -1 - \frac{l\kappa}{18\sigma H_0^2 e^{-H_0 l t} + 3\kappa}. \tag{5.17}$$

If $H_0 l > 0$ then as $t \to \infty$ we get again

$$\omega = -1 - \frac{l}{3}, \tag{5.18}$$

which is same as equation (5.14). This last equation tells us that our model can describes the accelerated expansion of the Universe e.g. for $l \geq 0$. Also it corresponds to the phantom case if $l > 0$. Finally we present other forms of the generalized Friedmann equations in the system (5.9). Let us rewrite these equations in the standard form as

$$3H^2 = \rho + \rho_z, \tag{5.19}$$

$$-(2\dot{H} + 3H^2) = p + p_z. \tag{5.20}$$

Here

$$\rho_z = 0.5\sigma^{-1}(\dot{a}z_{\dot{a}} - z), \tag{5.21}$$

$$p_z = -0.5\sigma^{-1}(\dot{a}z_{\dot{a}} - z) - \frac{1}{6\sigma}a(\dot{z}_{\dot{a}} - z_a) \tag{5.22}$$

are the z or $u - v$ contributions to the energy density and pressure, respectively.

6 Conclusion

In [30], Buchdahl proposed to replace the Einstein-Hilbert scalar Lagrangian R with a function of the scalar curvature. The resulting theory is nowadays known as $F(R)$ gravity. Almost 40 years later, Bengochea and Ferraro proposed to replace the TEGR that is the torsion scalar Lagrangian T with a function $F(T)$ of the torsion scalar, and studied its cosmological consequences [31]. This type of modified gravity is nowadays called as $F(T)$ gravity theory. These two gravity theories [that is $F(R)$ and $F(T)$] are, in some sense, alternative ways to modify GR. From these results arises the natural question: how we can construct some modified gravity theory which unifies $F(R)$ and $F(T)$ theories? Examples of such unified curvature-torsion theories were proposed in [10]-[11]. Such type of modified gravity theory is called the $F(R,T)$ gravity. In this $F(R,T)$ gravity, the curvature scalar R and the torsion scalar T play the same role and are dynamical quantities. In this paper, we have shown that the $F(R,T)$ gravity can be derived from the geometrical point of view. In particular, we have proposed a new method to construct particular models of $F(R,T)$ gravity. As an example we have considered the M_{43} - model, deriving its action in terms of the curvature and torsion scalars. Then in detail we have studied the M_{37} - model and presented its action, Lagrangian and equations of motion for the FRW metric case. Finally we have shown that the last model can describes the accelerated expansion of the Universe.

Concluding, we would like to note that in the paper, we present a special class of extended gravity models depending on arbitrary function $F(R,T)$, where R is the Ricci scalar and T the scalar torsion. While in the traditional Eistein-Cartan theory, the role of the torsion depends on the non trivial source associated with spin matter density, in our $F(R,T)$ gravity models, the torsion can propagate without the presence of spin matter density. In fact this is a crucial point, otherwise the additional scalar torsion degree of freedom are not different from the additional metric gravitational degree of freedom present in extended $F(R)$ models. Finally we would like to note that all results of this paper are new and different than results of our previous papers [10]-[11] on the subject.

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